Dynamics and Multiscale Modular Structure in Networks

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Imperial College London
100 years of living science

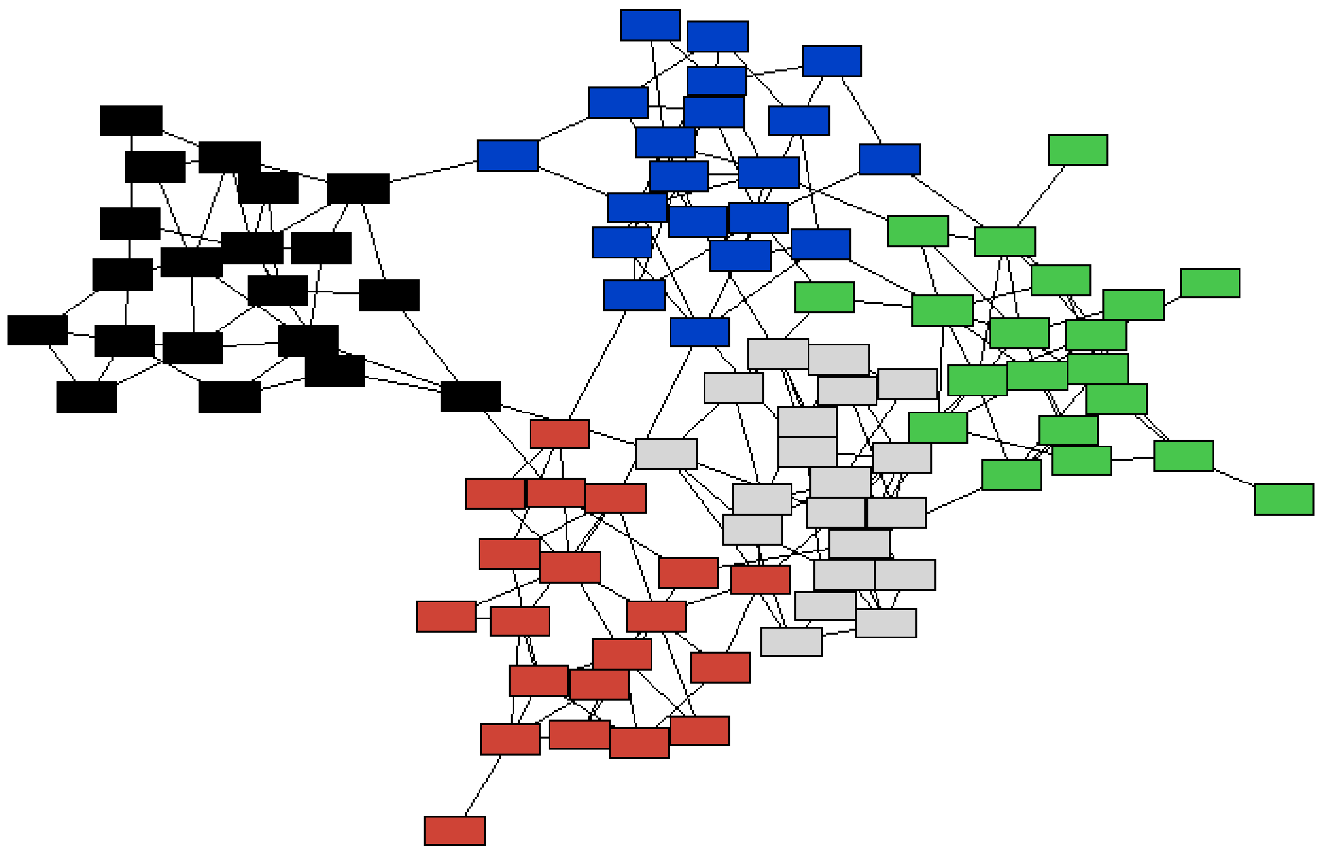
R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*

V.D. Blondel, J.-L. Guillaume, R. Lambiotte and E. Lefebvre, J. Stat. Mech., P10008 (2008).

1. Modules and Hierarchies
2. Quality of a partition: modularity and stability
3. Modularity optimisation
4. Selection of the most relevant scales

Modular Networks

Most networks are very inhomogeneous and are made of modules:
many links within modules and a few links between different modules



Internet

Power grids

Food webs

Metabolic networks

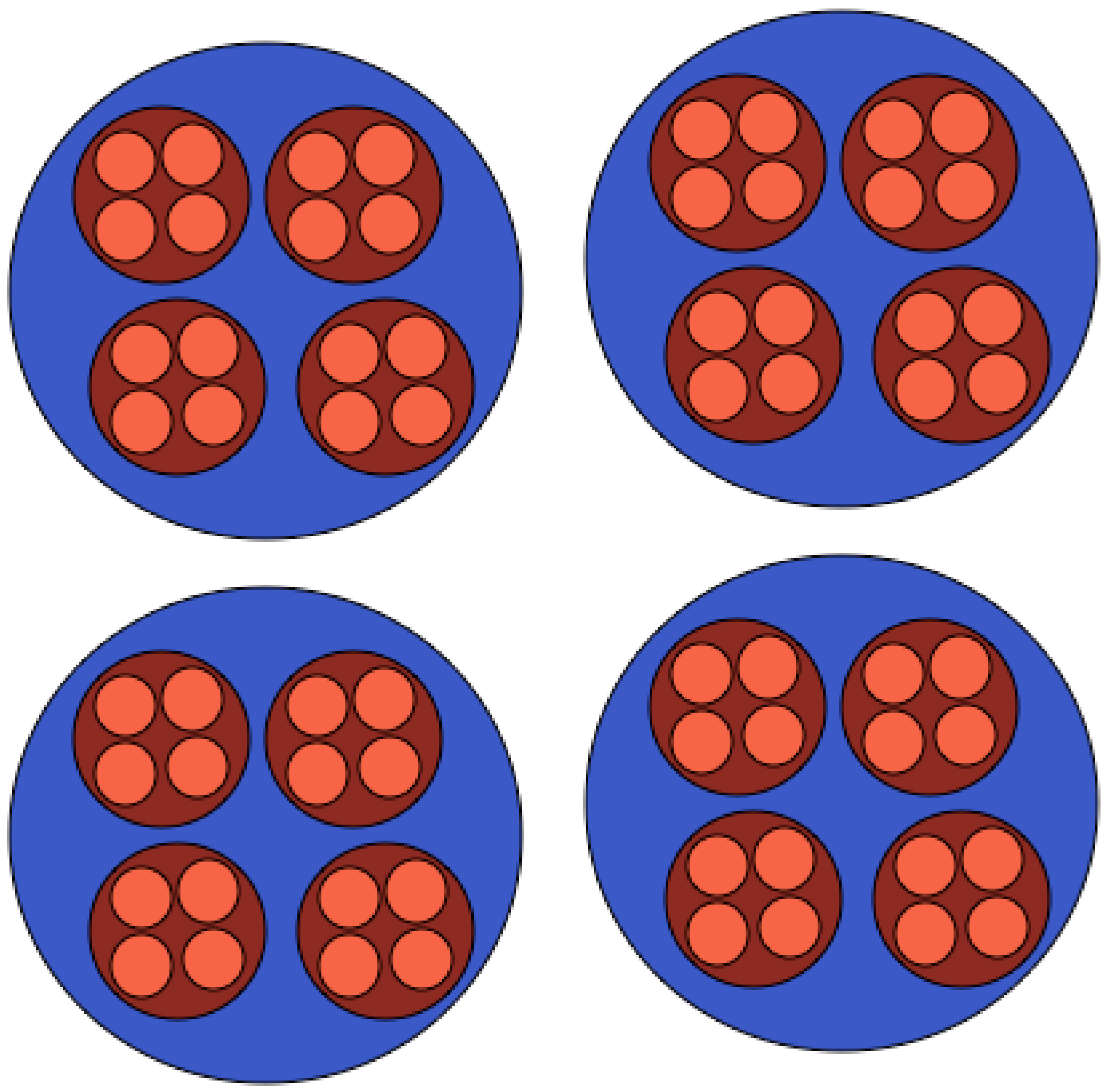
Social networks

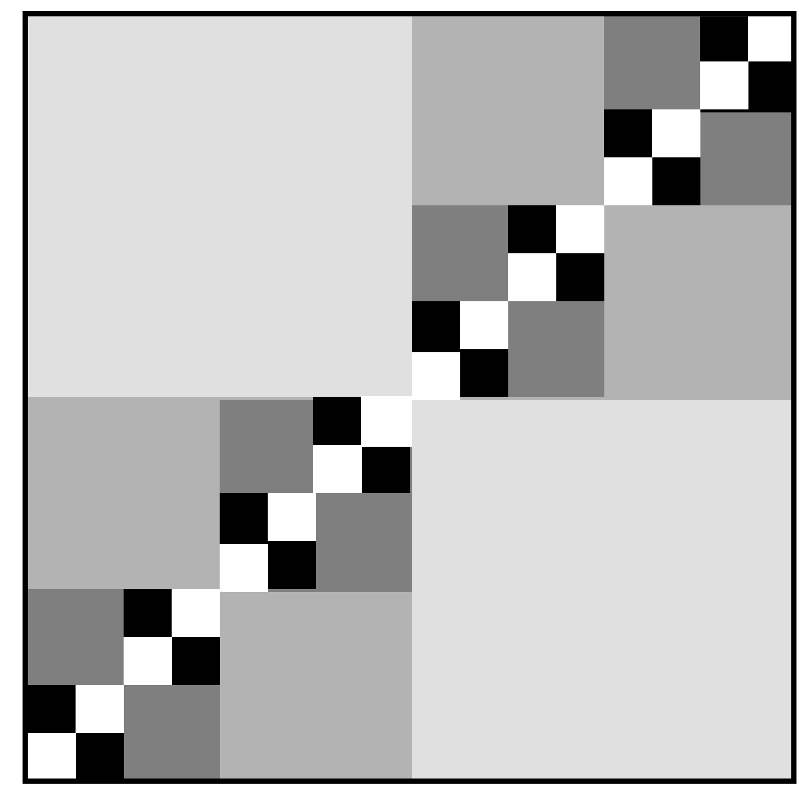
The brain

Etc.

Modular Networks

Networks have a hierarchical structure: modules within modules

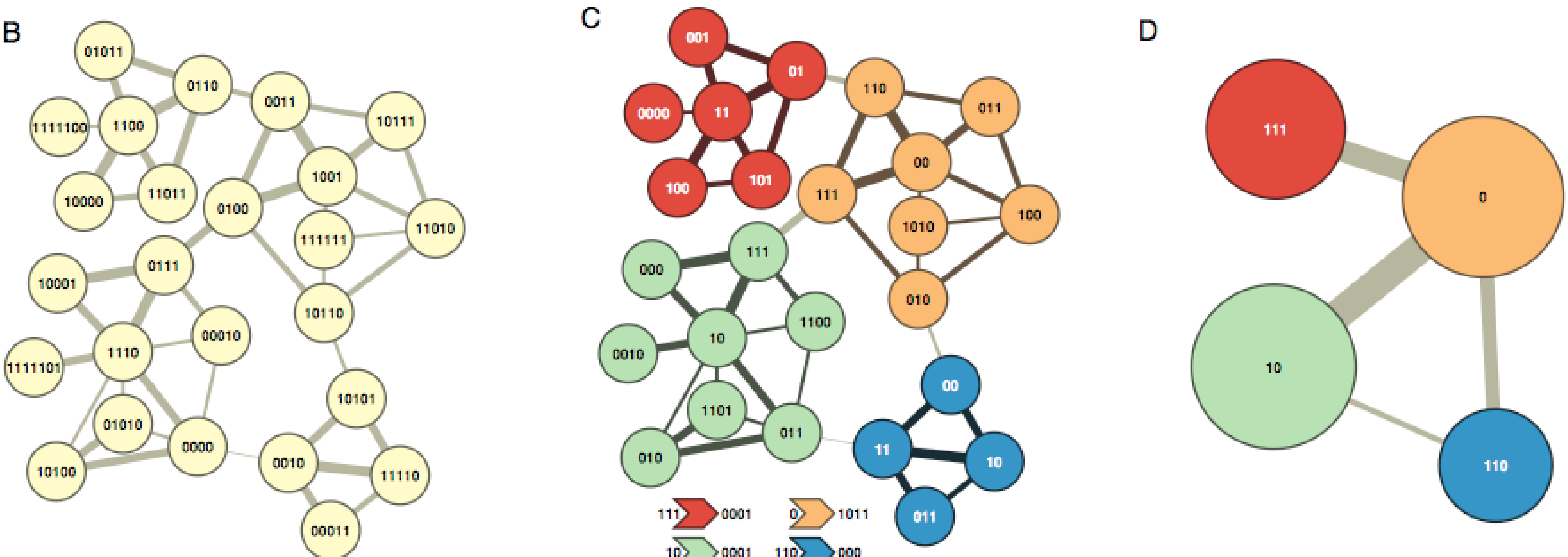




Simon, H. (1962). The architecture of complexity. Proceedings of the American Philosophical Society, 106, 467–482.

Modular Networks

Uncovering communities/modules helps to understand the structure of the network, to uncover similar nodes and to draw a readable map of the network (when N is large).



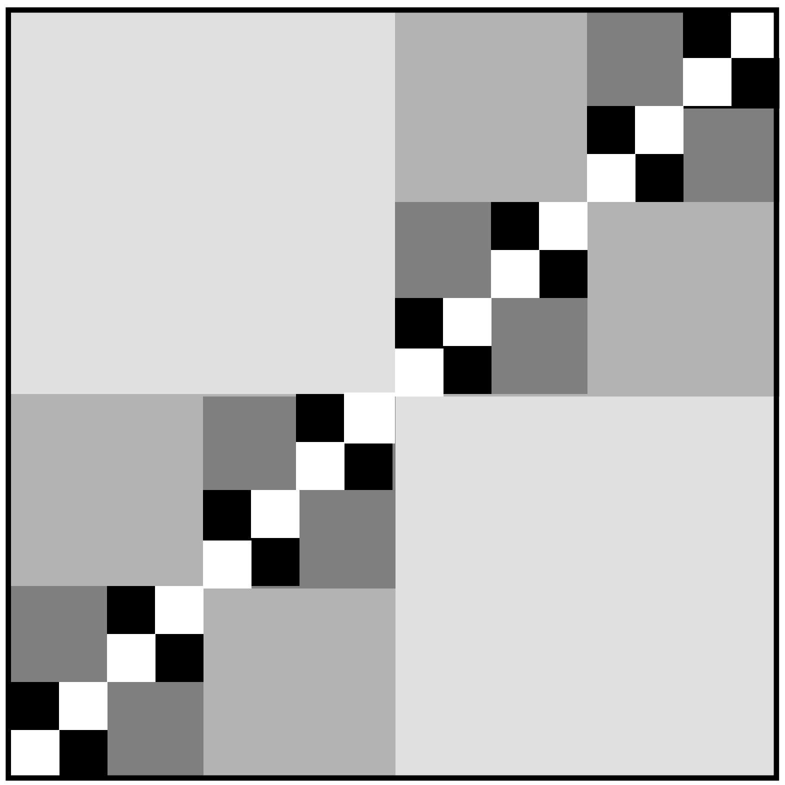
Find a partition of the network into communities

Coarse-grained description

Martin Rosvall and Carl T. Bergstrom, PNAS 105, 1118 –1123 (2008)

Modular Networks and dynamics

Many networks are “modular” and have a hierarchical structure:
modules within modules



How does such modularity affect dynamics?

A. Arenas, A. Diaz-Guilera and C.J. Pérez-Vicente, *Phys. Rev. Lett.* **96**, 114102 (2006).
R. Lambiotte, M. Ausloos and J.A. Holyst, *Phys. Rev. E* **75**, 030101(R) (2007).

Is it possible to uncover those modules in large networks?

NG, GN, Walktrap, clique-percolation, Simulated Annealing, etc.

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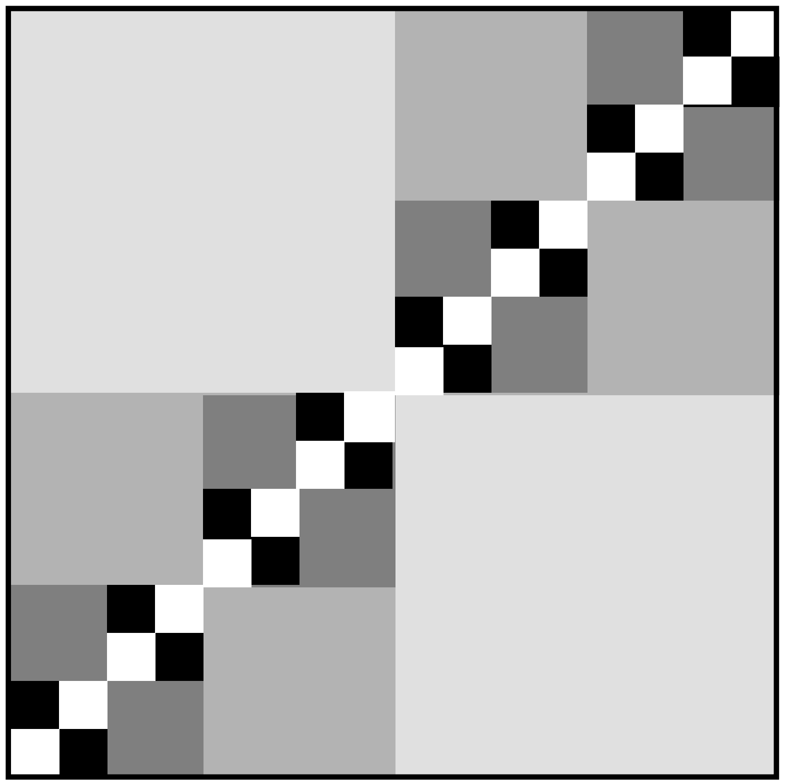

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Is it possible to use dynamics to characterize (and uncover?) the modular structure of a network?

e.g. Walktrap (RW exploration), Rosvall and Bergstrom, etc.

Is it possible to uncover those modules in large networks?

NG, GN, Walktrap, clique-percolation, Simulated Annealing, etc.

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Notations

Let us focus on an unweighted, undirected network

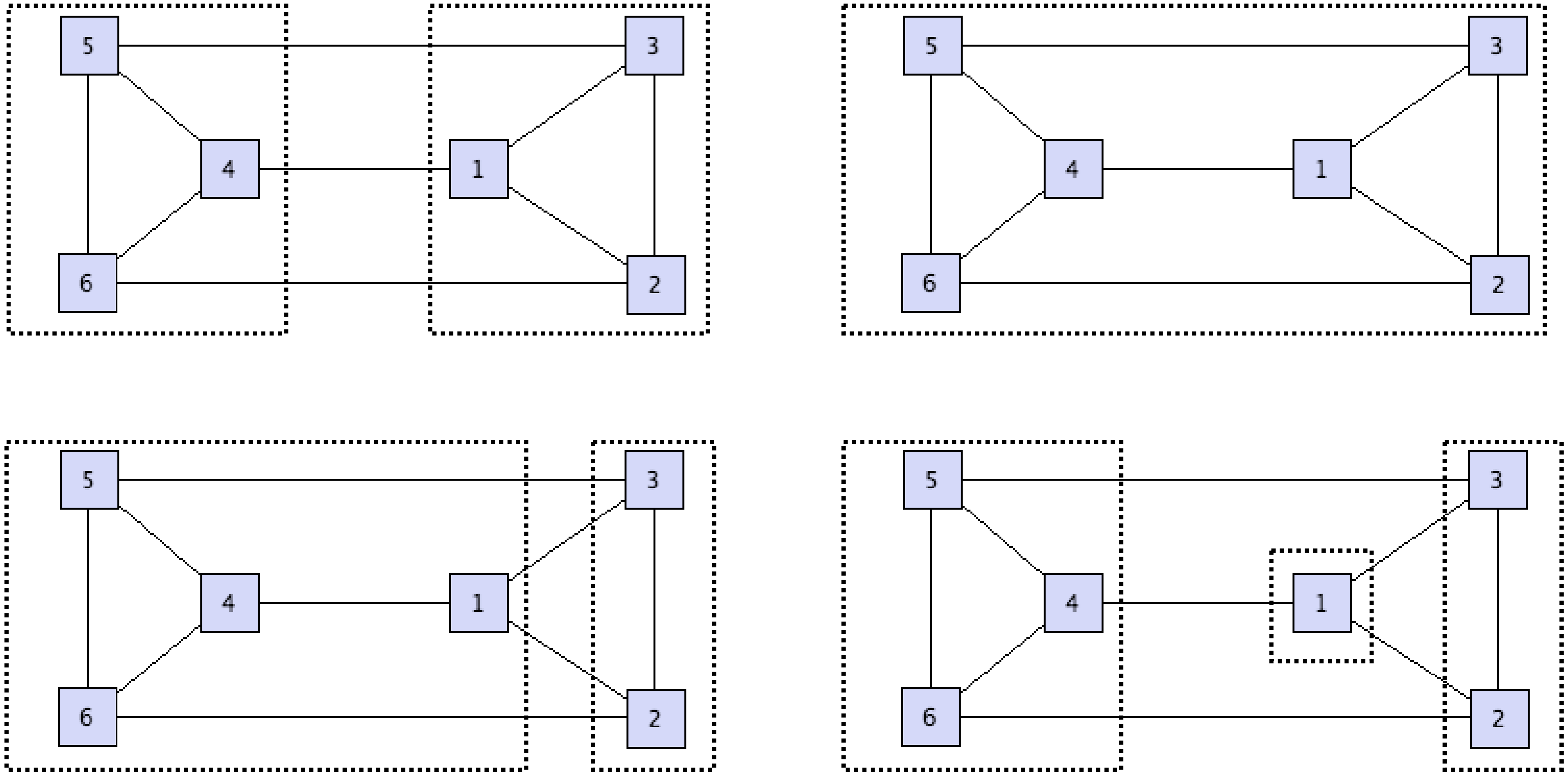
A_{ij} adjacency matrix

$k_i = \sum_j A_{ij}$ *degree* of *i*

$m = \frac{1}{2} \sum_i k_i$ total number of links

Quality of a partition

What is the best partition of a network into modules?



....

Modularity

Q = fraction of edges within communities - expected fraction of such edges

Let us attribute each node *i* to a community c_i

$$Q = \frac{1}{2m} \sum_{i,j} \left[A_{ij} - P_{ij} \right] \delta(c_i, c_j)$$

$$Q \in [-1, 1]$$

$$P_{ij} = \frac{k_i k_j}{2m}$$

expected number of links between i and j

$$\rightarrow \quad Q_C = \frac{1}{2m} \sum_{i,j} \left[A_{ij} - k_i k_j / 2m \right] \delta(c_i, c_j)$$

Modularity

Optimising modularity uncovers one partition

What about sub (or hyper)-communities in a hierarchical network?

Reichardt & Bornholdt

$$Q_\gamma = \frac{1}{2m}\sum_{i,j}\left[A_{ij} - \gamma P_{ij}\right]\delta(c_i, c_j)$$

Arenas et al.

$$Q(A_{ij} + rI_{ij})$$

Tuning parameters allow to uncover communities of different sizes

Reichardt & Bornholdt different of Arenas, except in the case of a regular graph where

$$\gamma = 1 + r/\langle k\rangle$$

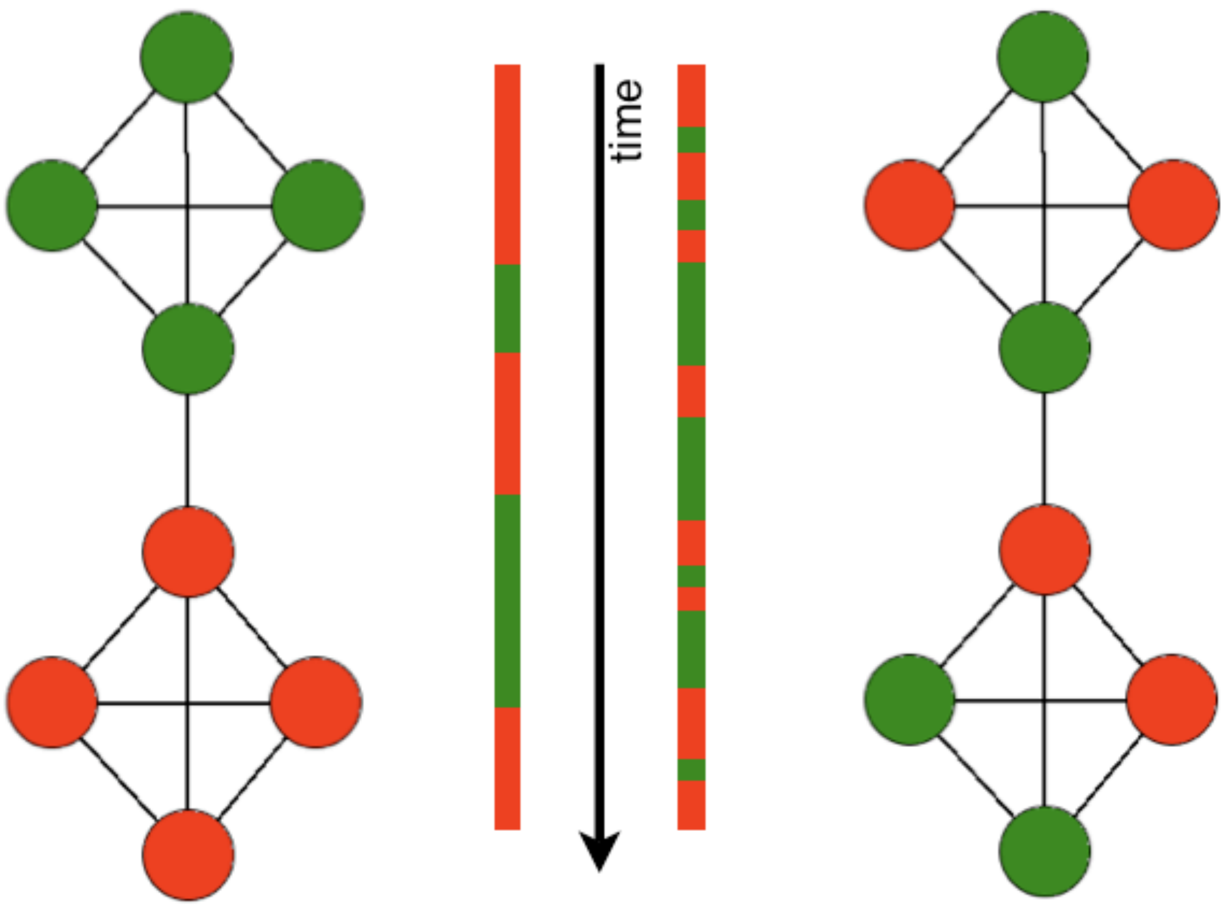
J. Reichardt and S. Bornholdt, Phys. Rev. E ***74****, 016110 (2006). Statistical mechanics of community detection*

A Arenas, A Fernandez, S Gomez, New J. Phys. ***10****, 053039 (2008). Analysis of the structure of complex networks at different resolution levels*

Stability

The quality of a partition is determined by the patterns of a flow within the network: a flow should be trapped for long time periods within a community before escaping it.

The stability of a partition is defined by the statistical properties of a random walker moving on the graph:



J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. arXiv:0812.1811.

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$$R(t) = \sum_{C \in \mathcal{P}} P(C, t_0, t_0 + t) - P(C, t_0, \infty)$$

$P(C, t_0, t_0 + t)$ probability for a walker to be in the same community at times t_0 and $t_0 + t$ when the system is at equilibrium

$P(C, t_0, \infty)$ probability for two independent walkers to be in C (ergodicity)

J.-C. Delvenne, S. Yaliraki & M. Barahona, *Stability of graph communities across time scales*. arXiv:0812.1811.

Stability

Let us consider a continuous-time random walk with Poisson waiting times

$$\dot{p}_i = \sum_j \frac{A_{ij}}{k_j} p_j - p_i \quad \xrightarrow{\text{equilibrium}} \quad p_i^* = k_i/2m$$

$$R(t) = \sum_{i,j} \left[\left(e^{t(B-I)}\right)_{ij} \frac{k_j}{2m} - \frac{k_i k_j}{(2m)^2} \right] \delta(c_i, c_j)$$

$$B_{ij} = A_{ij}/k_j$$

Probability that a walker is in the same community initially and at time t

Same probability for independent walkers

R. Lambiotte, J.-C. Delvenne and M. Barahona, *arXiv:0812:1770*

Stability: time as a resolution parameter

What are the optimal partitions of R_t?

t=0 $$R(0) = 1 - \sum_{i,j} \frac{k_i k_j}{(2m)^2} \delta(c_i, c_j)$$ → Communities=single nodes

t small $$R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$$

favours single nodes ↙ ↘ modularity

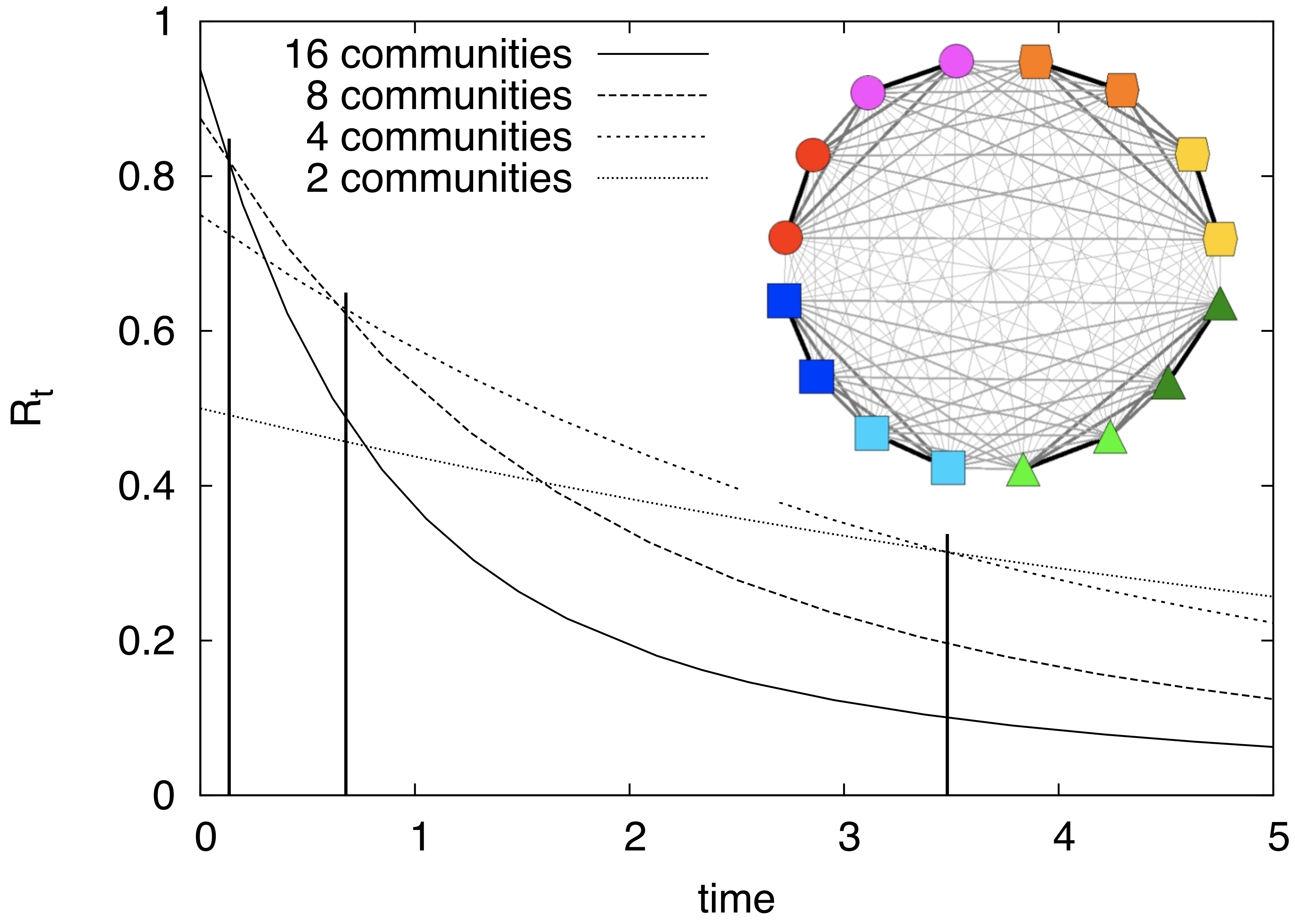
!! Q_t equivalent to the Hamiltonian formulation of Reichardt and Bornholdt (t=1/gamma)

....

When t goes to infinity, the optimal partition is made of 2 communities (by spectral decomposition)

Stability: time as a resolution parameter

Time is a “resolution parameter”: larger and larger communities when time is increased



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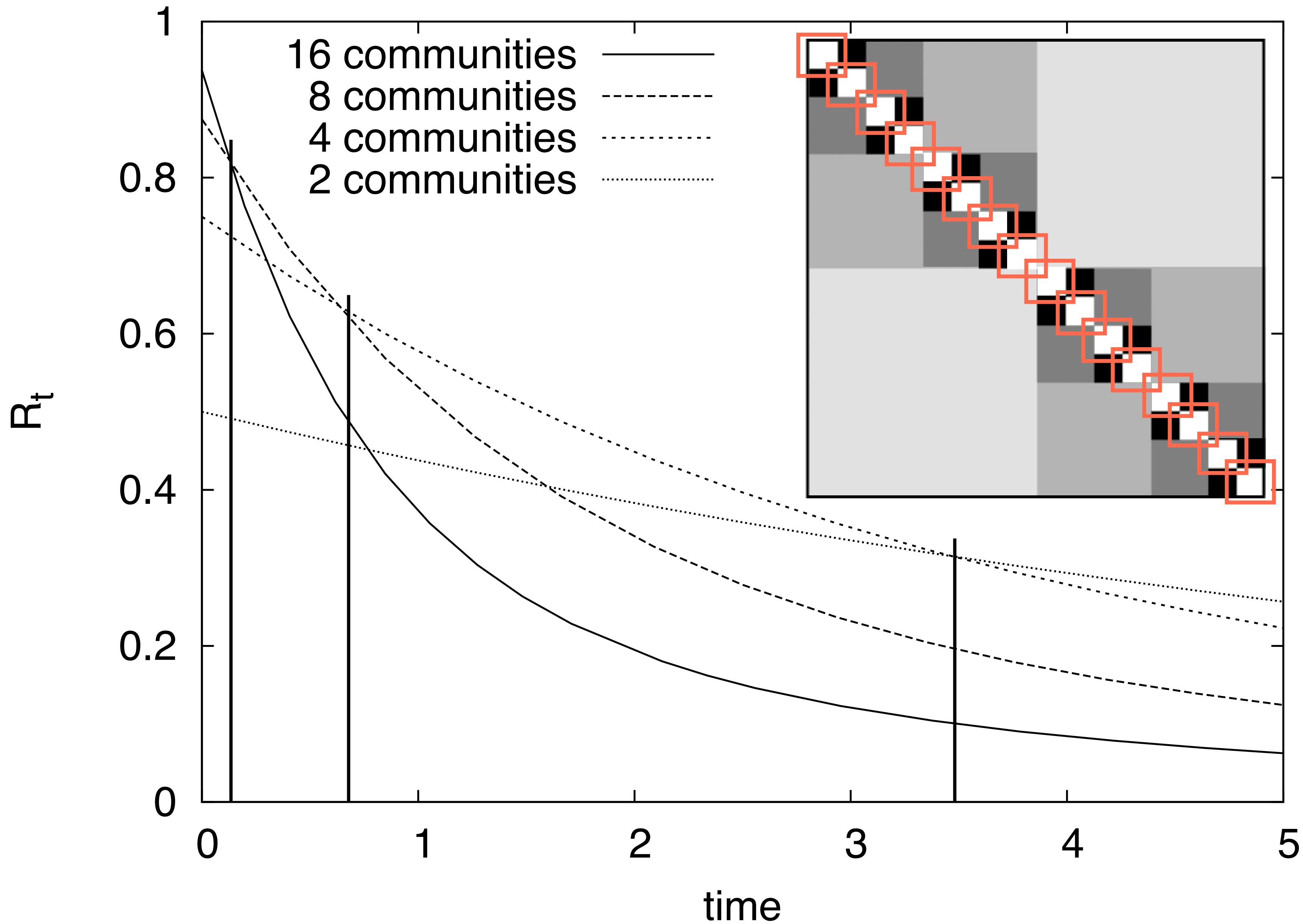

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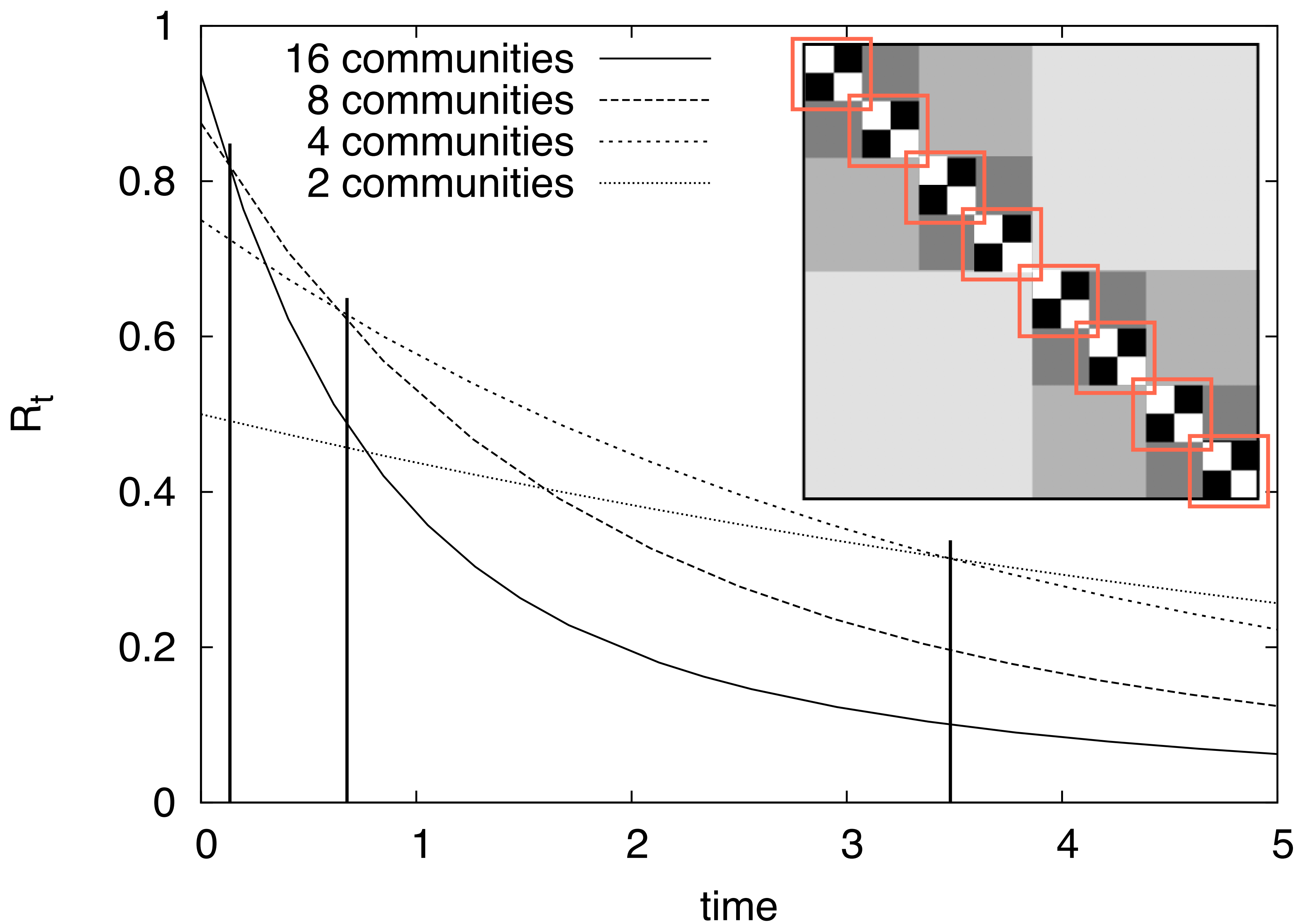

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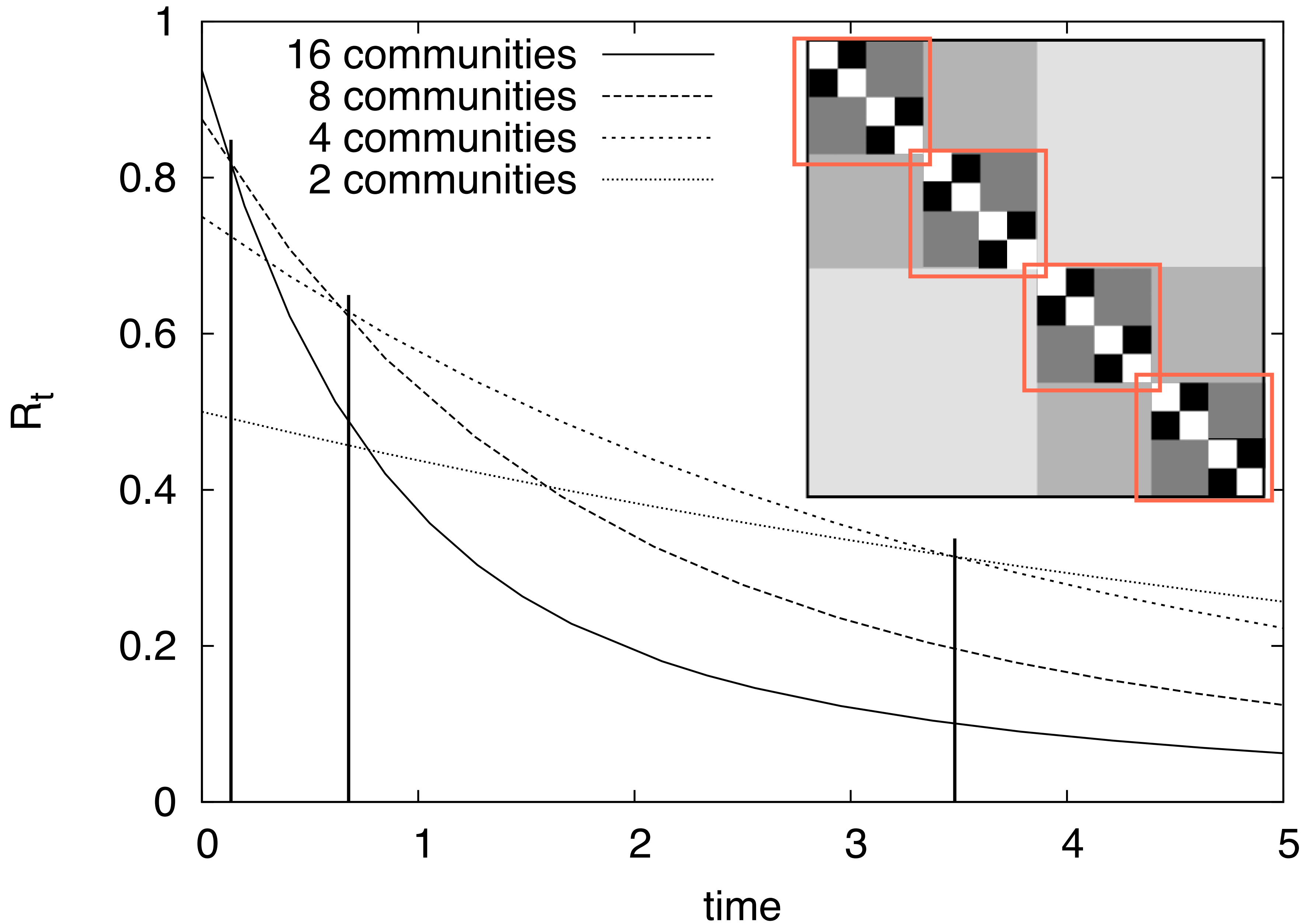

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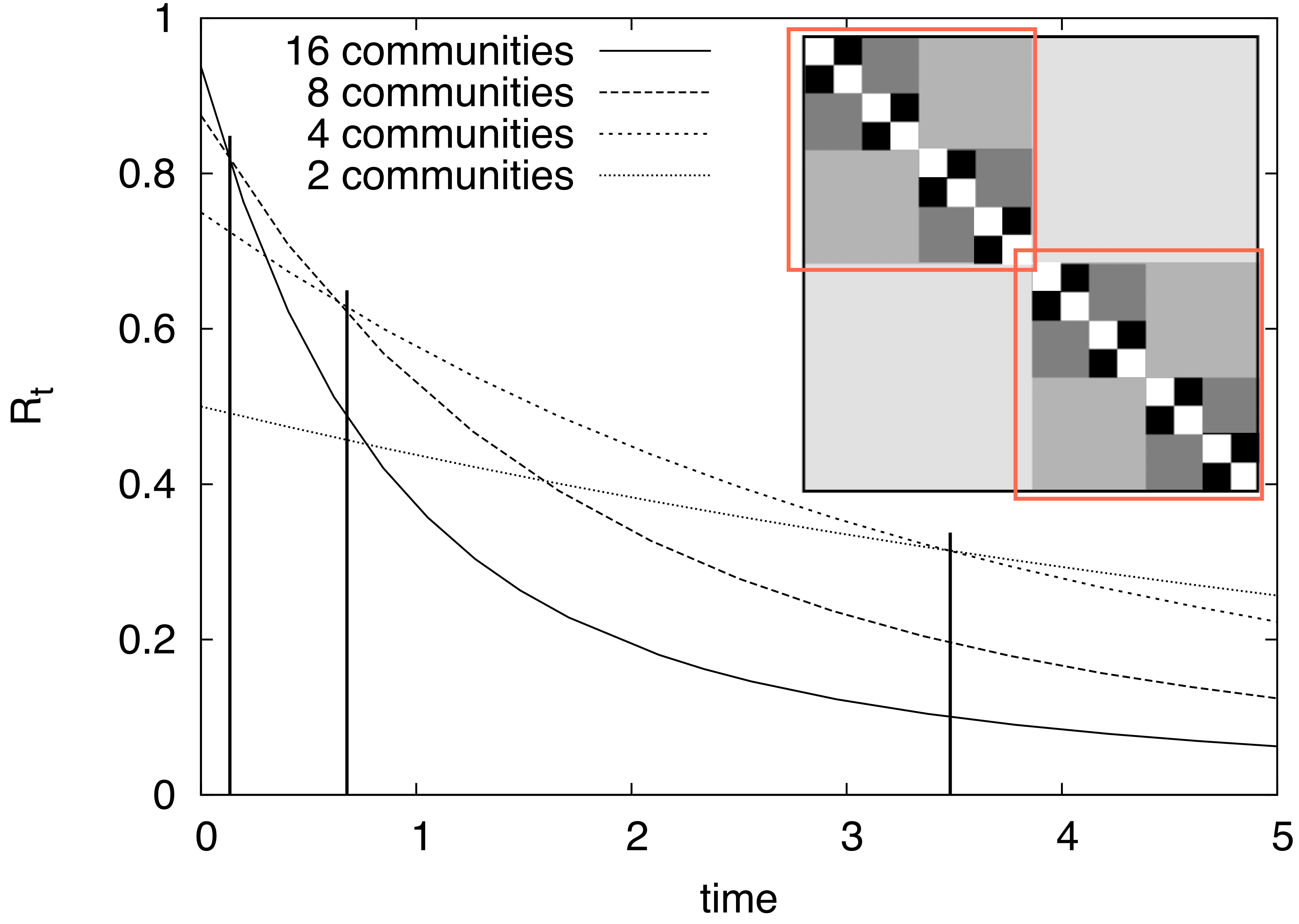

Important observation:

The stability *R(t)* of the partition of a graph with adjacency matrix *A* is equivalent to the modularity *Q* of a time-dependent graph with adjacency matrix *X(t)*

$$X_{ij}(t) = \left(e^{t(B-I)}\right)_{ij} k_j \qquad X_{ij}(t) = X_{ji}(t)$$

which is the flux of probability between 2 nodes at equilibrium and whose generalised degree is

$$\sum_j X_{ij}(t) = k_i$$

$$R(t) = \sum_{i,j} X_{ij}(t)/2m - k_i k_j/(2m)^2 \;\; \delta(c_i, c_j) = Q(X(t))$$

For very large networks: $R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$

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Modularity optimisation

Different types of algorithm for different applications:

Small networks (<10^3): Simulated Annealing

Intermediate size (10^3 -10^4): Spectral methods, PL, etc.

Large size (>10^5): greedy algorithms

Our algorithm

The algorithm is based on two steps that are repeated iteratively.

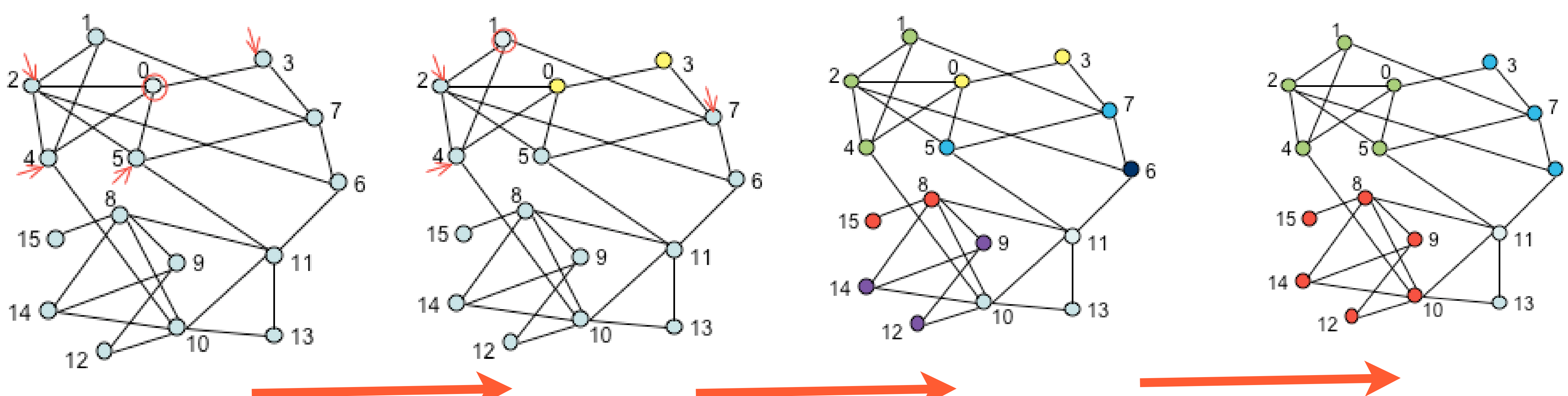
First phase: Find a local maximum

1) Give an order to the nodes (0,1,2,3,...., N-1)

2) Initially, each node belongs to its own community (N nodes and N communities)

3) One looks through all the nodes (from 0 to N-1) in an ordered way. The selected node looks among its neighbours and adopt the community of the neighbour for which the increase of the quality function is maximum (and positive).

4)This step is performed iteratively until a local maximum of modularity is reached (each node may be considered several times).



Node 0 moves to the community of Node 3

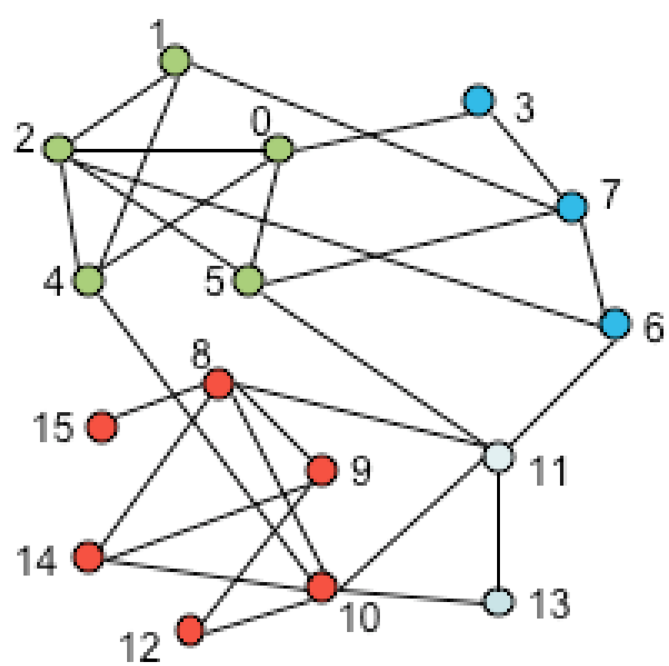
After N nodes have been considered

After each nodes has been considered 4 times

V.D. Blondel, J.-L. Guillaume, RL and E. Lefebvre, *J. Stat. Mech.*, P10008 (2008).

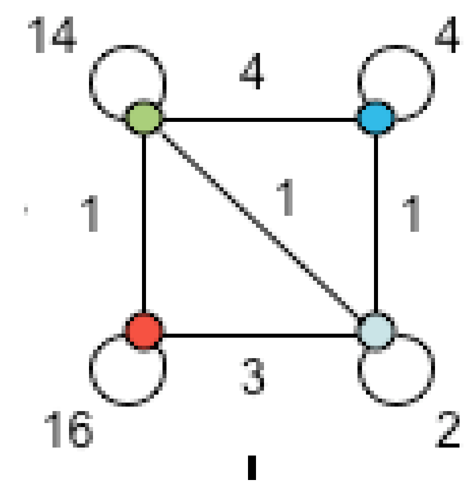
Our algorithm

Once a local maximum has been attained, we build a new network whose nodes are the communities. The weight of the links between communities is the total weight of the links between the nodes of these communities.



New network of 4 nodes!

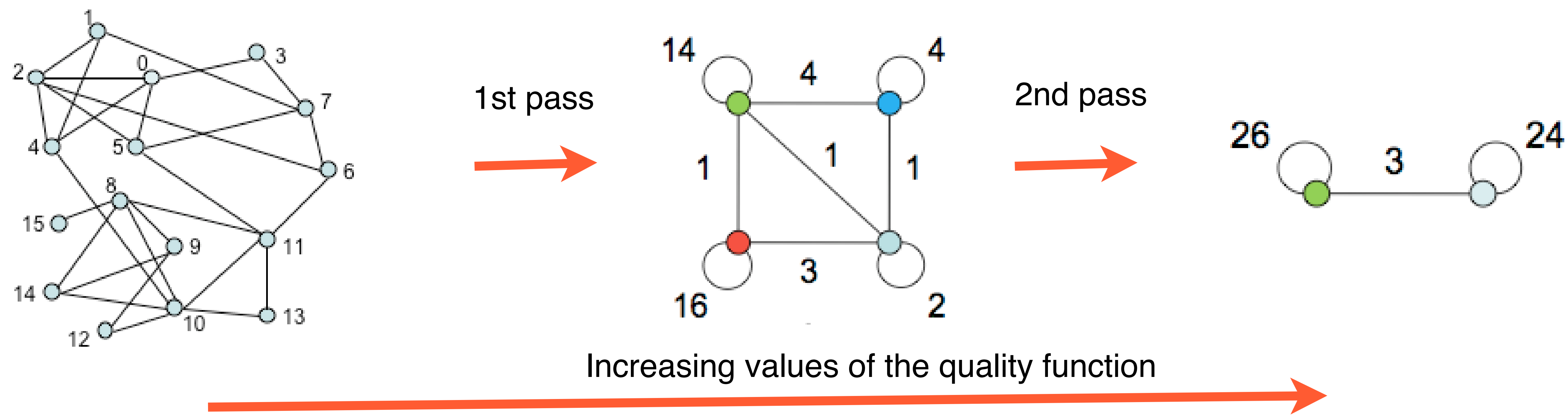
Note the self-loops



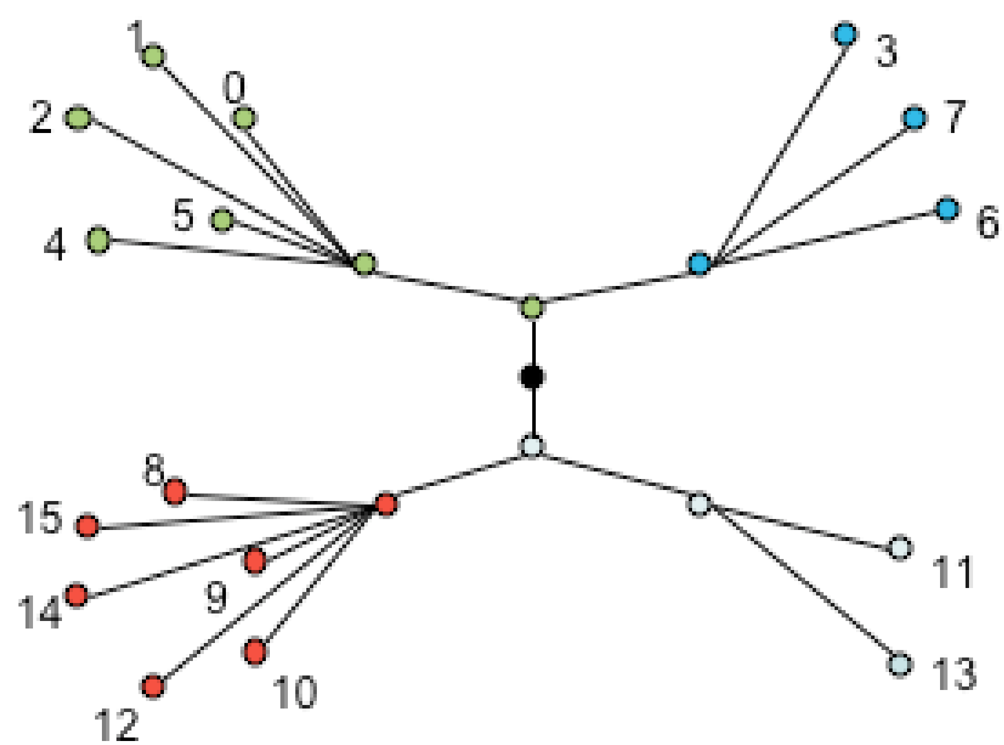
In typical realizations, the number of nodes diminishes drastically at this step.

Our algorithm

The two steps are repeated iteratively, thereby leading to a hierarchical decomposition of the network.



Hierarchical representation



Advantages of the Louvain method

Known to perform very well for optimising modularity:

	Karate	Arxiv	Internet	Web nd.edu	Phone	Web uk-2005	Web WebBase 2001
Nodes/links	34/77	9k/24k	70k/351k	325k/1M	2.04M/5.4M	39M/783M	118M/1B
CNM	.38/0s	.772/3.6s	.692/799s	.927/5034s	-/-	-/-	-/-
PL	.42/0s	.757/3.3s	.729/575s	.895/6666s	-/-	-/-	-/-
WT	.42/0s	.761/0.7s	.667/62s	.898/248s	.553/367s	-/-	-/-
Our algorithm	.42/0s	.813/0s	.781/1s	.935/3s	.76/44s	.979/738s	.984/152mn

Also works quite well for *Q(t)*.
E.g., football network:

$$R(t) \approx (1-t)R(0) + tQ_C \equiv Q(t)$$

SA, *iteration factor=1.0*; *cooling factor=0.995*

0.0 115 0.991245 115 23
0.1 115 0.894656 33 63
0.2 115 0.850513 14 58
0.3 115 0.817482 12 40
0.4 115 0.786487 12 36
0.5 115 0.755492 12 36
0.6 115 0.724497 12 51
0.7 115 0.693602 11 36
0.8 115 0.663422 11 33
0.9 115 0.633770 10 58
1.0 115 0.601357 9 79

Louvain method:

0.0 115 0.991245 115 0
0.1 115 0.894616 33 0
0.2 115 0.850513 14 0
0.3 115 0.817482 12 0
0.4 115 0.786487 12 0
0.5 115 0.755492 12 0
0.6 115 0.724497 12 0
0.7 115 0.693602 11 0
0.8 115 0.663422 11 0
0.9 115 0.633792 10 0
1.0 115 0.604570 10 0

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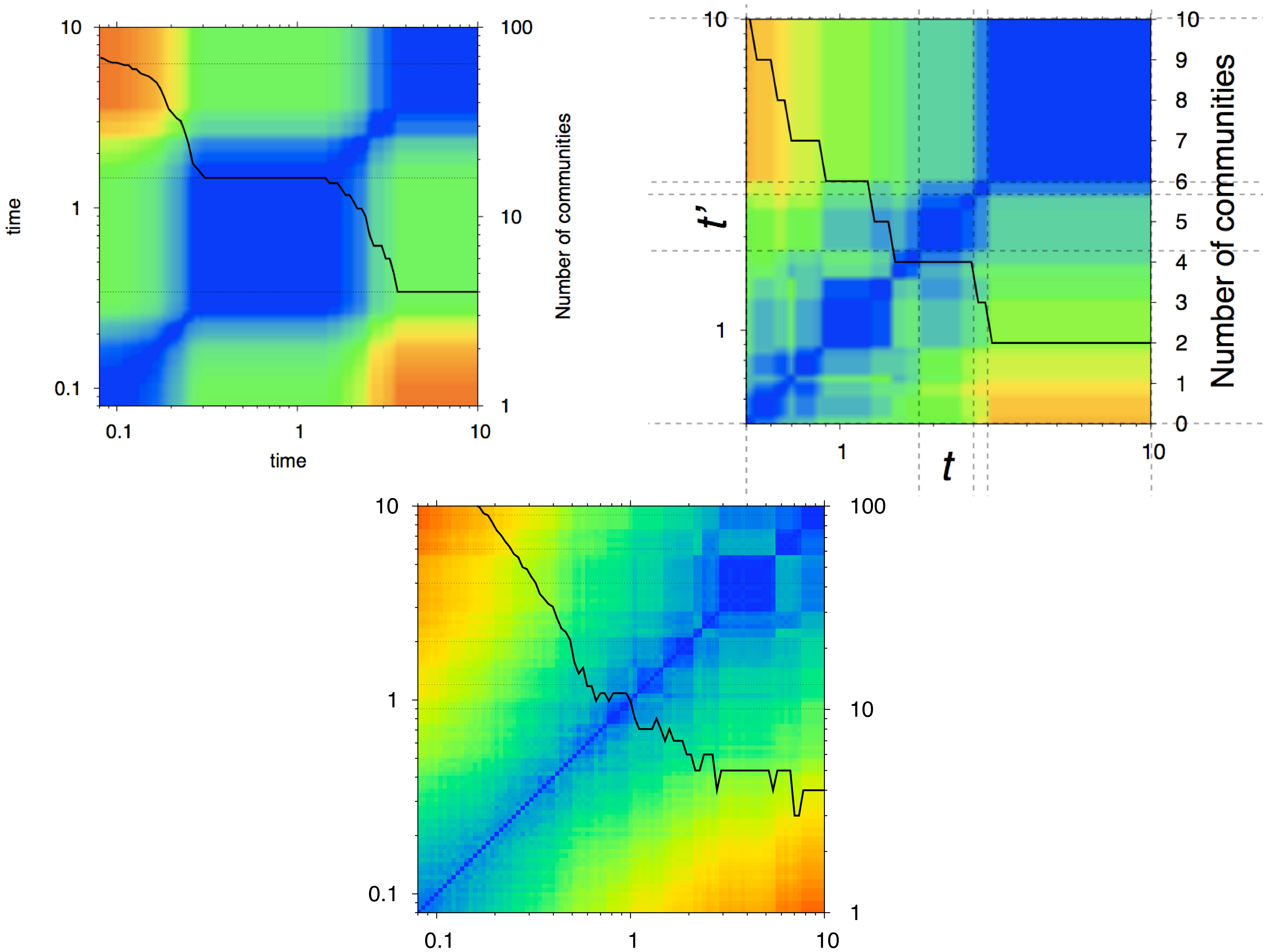
Selection of the most relevant scales

The optimization of *R(t)* over a period of time leads to a sequence of partitions that are optimal at different time scales.
How to select the most relevant scales of description?

The significance of a particular scale is usually associated to a certain notion of the robustness of the optimal partition. Here, robustness indicates that a small modification of the optimization algorithm, of the network, or of the quality function does not alter this partition.

We look for regions of time where the optimal partitions are very similar. The similarity between two partition is measured by the *normalised variation of information*.


time
time
Number of communities
t'
t
Number of communities

Conclusion

- Relation between dynamics and the hierarchical structure of networks
- Dynamical formulation for the quality of a partition
- Changing time allows to zoom in and out
- Algorithms developed in order to optimise stability/modularity very large networks

- Different dynamics lead to different quality functions for the partition of a graph
- Modularity and Stability are radically different in the case of directed networks

Original Louvain method to optimise modularity available on *http://findcommunities.googlepages.com*

Generalized codes to optimise Q_t available on http://*www.lambiotte.be*

Thanks to J.-L. Guillaume (for providing his c++ code)

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